

On the cohomology of pure mapping class groups as FI-modules

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Abstract

In this paper we apply the theory of finitely generated FI-modules developed by Church, Ellenberg and Farb to certain sequences of rational cohomology groups. Our main examples are the cohomology of the moduli space of n -pointed curves, the cohomology of the pure mapping class group of surfaces and some manifolds of higher dimension, and the cohomology of classifying spaces of some diffeomorphism groups. We introduce the notion of $\mathbf{FI}[G]$ -module and use it to strengthen and give new context to results on representation stability discussed by the author in a previous paper. Moreover, we prove that the Betti numbers of these spaces and groups are polynomial and find bounds on their degree. Finally, we obtain rational homological stability of certain wreath products.

1 Introduction

Our objective is to study some examples of sequences of cohomology groups that have an underlying structure of an FI-module as defined by Church–Ellenberg–Farb in [CEF] and derive as many consequences as we can from this approach.

We denote by \mathbf{FI} be the category of finite sets and injections. First we will consider a functor X from \mathbf{FI}^{op} to the category \mathbf{Top} of topological spaces or to the category \mathbf{Gp} of groups. In the first case we call X a *co-FI-space*, in the second case X is a *co-FI-group*. Given such a functor X , for any $i \geq 0$, we will be interested in the *FI-module* $H^i(X)$ over \mathbb{Q} . This is a functor from \mathbf{FI} to $\mathbf{Vect}_{\mathbb{Q}}$, the category of \mathbb{Q} -vector spaces, that we obtain by composing X with the cohomology functor $H^i(-; \mathbb{Q})$. We can consider the graded version: $H^*(X)$ is called *graded FI-module*. In Section 2.1 below we recall the general definition of FI-module and some of its properties.

In this paper we will focus in the FI-modules that arise from the following examples of co-FI-spaces and co-FI-groups.

- (1) The co-FI-space M^\bullet . Given M a topological space, this is the functor from \mathbf{FI}^{op} to \mathbf{Top} given by $\mathbf{n} \mapsto M^n := \underbrace{M \times \cdots \times M}_n$. Similarly if G is a group, we can consider the co-FI-group G^\bullet .
- (2) The co-FI-space $\mathbf{Conf}_\bullet(M)$. For each $n \geq 1$ we consider the *configuration space of ordered n -tuples of distinct points on a topological space M* :

$$\mathbf{Conf}_n(M) := \{(x_1, \dots, x_n) \in M^n : x_i \neq x_j \text{ if } i \neq j\}.$$

The co-FI-space $\mathbf{Conf}_\bullet(M)$ is given by $\mathbf{n} \mapsto \mathbf{Conf}_n(M)$.

- (3) The co-FI-space $\mathcal{M}_{g,\bullet}$. Let $g \geq 0$ and $n \geq 0$. We denote by $\mathcal{M}_{g,n}$ the moduli space of genus g Riemann surfaces with n marked points. The functor $\mathcal{M}_{g,\bullet}$ is given by $\mathbf{n} \mapsto \mathcal{M}_{g,n}$.

- (4) The co-FI-space $\overline{\mathcal{M}}_{g,\bullet}$. This is the functor $\mathbf{n} \mapsto \overline{\mathcal{M}}_{g,n}$, the Deligne-Mumford compactification of the moduli space $\mathcal{M}_{g,n}$.
- (5) The co-FI-group $\mathrm{PMod}^\bullet(M)$. Given M a connected, smooth manifold and let p_1, \dots, p_n be distinct points in the interior of M , we define the *mapping class group* to be the group

$$\mathrm{Mod}^n(M) := \pi_0(\mathrm{Diff}^n(M))$$

where $\mathrm{Diff}^n(M)$ is the subgroup of diffeomorphisms in $\mathrm{Diff}(M \text{ rel } \partial M)$ that leave invariant the set of points $\{p_1, \dots, p_n\}$. Similarly, we let $\mathrm{PDiff}^n(M)$ be the subgroup of diffeomorphisms in $\mathrm{Diff}(M \text{ rel } \partial M)$ that fix the points p_1, \dots, p_n pointwise and define the *pure mapping class group* as the group

$$\mathrm{PMod}^n(M) := \pi_0(\mathrm{PDiff}^n(M)).$$

The co-FI-group $\mathrm{PMod}^\bullet(M)$ is given by $\mathbf{n} \mapsto \mathrm{PMod}^n(M)$.

When $M = \Sigma_{g,r}$ is a compact orientable surface of genus $g \geq 0$ with $r \geq 0$ boundary components, we restrict to orientation-preserving self-diffeomorphisms. We denote $\mathrm{PMod}^\bullet(\Sigma_{g,r})$ by $\mathrm{PMod}_{g,r}^\bullet$.

- (6) The co-FI-space $B\mathrm{Diff}^\bullet(M)$. This is the functor $\mathbf{n} \mapsto B\mathrm{PDiff}^n(M)$, where $B\mathrm{PDiff}^n(M)$ is the classifying space of the group $\mathrm{PDiff}^n(M)$ defined in example (5) above. If M is orientable, we can restrict to orientation-preserving diffeomorphisms.

Remark: Observe that for each of the previous examples, the FI-module $H^i(X)$ encodes the information of the sequence $\{H^i(X_n; \mathbb{Q})\}_{n \in \mathbb{N}}$ of finite dimensional representations of the symmetric groups S_n .

1.1 What is known

Let X be any of the co-FI-spaces or co-FI-groups from examples (1), (2), (3), (5) and (6) above. A combination of the work by Church ([Chu]), Church–Ellenberg–Farb ([CEF]) and myself ([JR11]) implies that, under some hypotheses recalled below, for any $i \geq 0$ the FI-module $H^i(X)$ is *finitely generated* (see Section 2.1 for definition of finite generation). Church–Ellenberg–Farb proved that finite generation implies the following results.

For n sufficiently large with respect to i :

- (i) The character of the S_n -representation $H^i(X_n; \mathbb{Q})$ is of the form

$$\chi_{H^i(X_n; \mathbb{Q})} = Q_i(Z_1, Z_2, \dots, Z_r), \tag{1}$$

where $r > 0$ only depends on i and Q_i is a polynomial in the class functions

$$Z_k(\sigma) := \# \text{ cycles of length } k \text{ in } \sigma, \quad \text{for any } \sigma \in S_n.$$

Moreover the polynomial Q_i has bounded degree and the bound is independent of n (each Z_k has degree k). In particular, the dimension $\dim_{\mathbb{Q}}(H^i(X_n; \mathbb{Q})) = \chi_{H^i(X_n; \mathbb{Q})}(\mathrm{id})$ is a polynomial in n of bounded degree.

- (ii) The length of the representation $\ell(H^i(X_n; \mathbb{Q}))$ is bounded above independently of n . This gives a constraint on the structure of the S_n -representations in the decomposition into irreducibles (see Section 2.3 for a precise definition).

- (iii) The sequence of cohomology groups $\{H^i(X_n; \mathbb{Q})\}_n$ satisfies *uniform representation stability* in the sense of [CF] and monotonicity as defined in [Chu].
- (iv) The dimension $\dim_{\mathbb{Q}}(H^i(X_n/S_n; V(\lambda)_n))$ is eventually constant; here $V(\lambda)_n$ denotes the irreducible S_n -representation corresponding to the padded partition $\lambda[n]$ as defined below in Section 2.3.

A non-example: We mention the co-FI-space $\overline{\mathcal{M}}_{g,\bullet}$ in (4) above to exhibit an example of an FI-module that is not finitely generated. For instance $H^2(\overline{\mathcal{M}}_{g,\bullet})$ is not finitely generated for any $g > 2$ since computations of Arbarello and Cornalba in [AC96] show that for that case

$$\dim_{\mathbb{Q}}(H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})) = 2^{n-1}(g+1),$$

thus violating (i) above.

1.2 Main results

Table 1: Specific bounds

Group or Space	Sequence V_n	Stable Range	$\ell(V_n) \leq$	degree of $\chi_{V_n} \leq$	Stability Type
$M^n := \overbrace{M \times \cdots \times M}^n$	$H^i(M^n; \mathbb{Q})$ [CEF, PROOF OF THM 4.1]	$n \geq 2i$	$i + 1$	i	$(0, i)$
$\text{Conf}_n(M)$ $\dim(M) = d \geq 3$	$H^i(\text{Conf}_n(M); \mathbb{Q})$ [CEF, THM 4.2]	$n \geq 2i$	$i + 1$	i	$(i + 2 - d, i)$ $(0, i)$ if $\partial M \neq \emptyset$
$\text{Conf}_n(\Sigma)$ $\dim(\Sigma) = 2$	$H^i(\text{Conf}_n(\Sigma); \mathbb{Q})$ THEOREM 1.1	$n \geq 4i$ $n \geq 5i$	$2i + 1$	$2i$	$(2i, 2i)$ if Σ closed $(0, 2i)$ if $\partial \Sigma \neq \emptyset$ $(3i - 1, 2i)$ otherwise
$\mathcal{M}_{g,n}$ with $g \geq 2$	$H^i(\mathcal{M}_{g,n}; \mathbb{Q})$ THEOREM 1.2	$n \geq 6i$	$2i + 1$	$2i$	$(4i, 2i)$
$\text{PMod}_{g,r}^n$ $2g + r > 2$ and $r > 0$	$H^i(\text{PMod}_{g,r}^n; \mathbb{Q})$ THEOREM 1.2	$n \geq 4i$	$2i + 1$	$2i$	$(0, 2i)$
$\text{PMod}^n(M)$ $\dim M \geq 3$	$H^i(\text{PMod}^n(M); \mathbb{Q})$ THEOREM 1.3	$n \geq 3i$ $n \geq 2i$	$i + 1$	i	$(2i, i)$ $(0, i)$ if $\partial M \neq \emptyset$
$B\text{Diff}^n(M)$ $\dim M \geq 3$	$H^i(B\text{Diff}^n(M); \mathbb{Q})$ THEOREM 1.4	$n \geq 3i$	$i + 1$	i	$(2i, i)$

In this paper we revisit these examples and in some cases give direct proofs of finite generation in order to get better bounds on the degrees of the polynomials Q_i and the lengths of the representations. Moreover we are able to specify how large n needs to be so that the properties above are satisfied. Finally we improved the stable ranges for uniform representation stability given in [JR11]. The precise bounds obtained in each case are summarized in Table 1.

Remarks:

- (a) The explicit bounds for the first two examples in Table 1 are due to Church–Ellenberg–Farb. We recall them here as they are used as main ingredients in our proofs below. For each n , the cohomology ring $H^*(M^n; \mathbb{Q})$ is completely understood and $H^*(\text{Conf}_n(M); \mathbb{Q})$ is described by Totaro in [Tot96]. However, few explicit computations are known for the other examples in Table 1.
- (b) Notice that the injectivity degree and the stable range can be improved for the mapping class groups of manifolds with nonempty boundary. This is proved in Proposition 6.1 below.
- (c) In each example, the upper bound d on degree of χ_{V_n} does not depend on n . This implies that for each $\sigma \in S_n$, the character $\chi_{V_n}(\sigma)$ only depends on the cycles of σ of length $\leq d$.

In what follows we describe our results more precisely. The definitions of *stable range*, *weight* and *stability type of an FI-module* are given in Section 2.

Cohomology of configuration spaces. We recall examples (1) and (2) in Section 4 as ingredients for the other examples. In the case of configuration spaces for manifolds of dimension 2 we give a direct proof of finite generation in order to compute the bounds mentioned in Table 1.

Theorem 1.1. *Let Σ be a connected, oriented manifold of dimension 2. For any $i \geq 0$, the FI-module $H^i(\text{Conf}_\bullet(\Sigma))$ is finitely generated of weight $\leq 2i$ and has stability type at most $(2i, 2i)$ when Σ is a closed surface, at most $(0, 2i)$ when $\partial\Sigma$ is nonempty, and at most $(3i - 1, 2i)$ otherwise.*

This result recovers the same stable range of $n \geq 4i$ obtained in [Chu, Theorem 1] for the case when Σ is a closed surface or has non-empty boundary.

Cohomology of $\mathcal{M}_{g,n}$, $\text{PMod}^n(M)$ and $B\text{PDiff}^n(M)$. For examples (3) and (5) we give direct proofs of finite generation and obtain the following specific bounds (see Section 6).

Theorem 1.2. *For any $i \geq 0$ and $2g + r > 2$ the FI-module $H^i(\text{PMod}_{g,r}^\bullet)$ is finitely generated of weight $\leq 2i$ and stability type at most $(4i, 2i)$.*

The moduli space $\mathcal{M}_{g,n}$ is a rational model for the classifying space $B\text{PMod}_g^n$ for $g \geq 2$ (see for example [HL97] or [Har88]). Hence

$$H^*(\mathcal{M}_{g,n}; \mathbb{Q}) \approx H^*(\text{PMod}_g^n; \mathbb{Q}). \quad (2)$$

Therefore, for any $i \geq 0$ the FI-module $H^i(\mathcal{M}_{g,\bullet})$ is precisely $H^i(\text{PMod}_g^\bullet)$ and Theorem 1.2 gives the corresponding bounds for this example. The previous theorem implies that the corresponding consistent sequences of rational S_n -representations $\{H^i(\text{PMod}_{g,r}^n; \mathbb{Q})\}$ and $\{H^i(\mathcal{M}_{g,n}; \mathbb{Q})\}$ satisfy uniform representation stability with stable range $n \geq 4i$ when $r > 0$ and $n \geq 6i$ in general. This improves the stable range

$$n \geq \min\{4i + 2(4g - 6)(4g - 5), 2i^2 + 6i\}$$

obtained in [JR11, Theorem 1.1].

Theorem 1.3. *Let M be a smooth connected manifold of dimension $d \geq 3$ such that $\pi_1(M)$ is of type FP_∞ (e.g. M compact). Suppose that $\pi_1(M)$ has trivial center or that $\text{Diff}(M)$ is simply connected. If the group $\text{Mod}(M)$ is of type FP_∞ , then for any $i \geq 0$ the FI-module $H^i(\text{PMod}^\bullet(M))$ is finitely generated of weight $\leq i$ and with stability type at most $(2i, i)$.*

Remark: In [Put] Putman defined the notion of *central stability*. With his method, he proves that when M is a smooth connected manifold of dimension $d \geq 2$ with nonempty boundary the sequence $H_i(\text{PMod}^n(M); \mathbb{Q})$ satisfies central stability. For fields this condition agrees with the structure of a finitely generated FI-module (see [CEF, Introduction]). His result covers some cases that we do not, although he obtains stable ranges that are exponential in i in contrast with our linear ranges.

In a similar way we consider example (6).

Theorem 1.4. *Let M be a connected real manifold of dimension $d \geq 3$. Suppose that the classifying space $B\text{Diff}(M \text{ rel } \partial M)$ has the homotopy type of a CW-complex with finitely many cells in each dimension. Then, for any $i \geq 0$, the FI-module $H^i(B\text{PDiff}^\bullet M)$ is finitely generated of weight $\leq i$ and it has stability type at most $(2i, i)$.*

In particular, Theorems 1.3 and 1.4 apply to irreducible, compact, orientable 3-manifolds M with nonempty boundary satisfying conditions (i)-(iv) in [HM97, Section 3].

Our main contribution in these cases is that we are able to get linear bounds in i for the degree of the polynomials Q_i and the lengths of the representations. Moreover our new stable ranges for uniform representation stability are also linear in i , instead of the quadratic bounds in i that were obtained in [JR11].

A unified approach. In this paper we develop a unified approach to proving finite generation for FI-modules that arise as in the examples above. In Section 3 we present a general spectral sequence argument that allows us to prove finite generation for examples (2), (3), (5) and (6) in Theorems 1.1, 1.2, 1.3 and 1.4. Furthermore, this approach applies to spectral sequences arising from “FI-fibrations” over a fixed space and “FI-group extensions” of a given group (see Section 5.2).

The basic idea is to use a spectral sequence of FI-modules converging to the graded FI-module of interest. We then use knowledge about finite generation of the FI-modules in the E_2 -page and an inductive process together with closure properties of finite generation under subquotients and extensions to get our conclusion. The main difference between Theorems 1.1 and the other theorems are the type of spectral sequence that we use and the way that finite generation is proved for the E_2 -page.

FI[G]-modules. Let G be a group. In Section 5 we introduce the notion of an FI[G]-module: it is a functor V from the category **FI** to the category **G-Mod** of G -modules over \mathbb{Q} . This definition incorporates the action of a group G on our sequences of S_n -representations and allows us to take V as twisted coefficients for cohomology. For X , a path-connected space with fundamental group G , and $p \geq 0$, we are interested in the FI-module $H^p(X; V)$ given by $\mathbf{n} \mapsto H^p(X; V_n)$. Our major result in Section 5 uses finite generation of an FI[G]-module V to obtain finite generation and specific bounds for the new FI-modules $H^p(X; V)$.

Theorem 1.5 (Cohomology with coefficients in a f.g. FI[G]-module). *Let G be the fundamental group of a connected CW complex X with finitely many cells in each dimension. Consider a finitely generated FI[G]-module V over \mathbb{Q} of weight $\leq m$ with stability degree $\leq N$. Then for every $p \geq 0$, the FI[G]-module $H^p(X; V)$ is finitely generated of weight $\leq m$ with stability degree N .*

This is our tool to prove the base of the induction in the spectral sequence argument for Theorems 1.2, 1.3 and 1.4 above.

Manifolds with boundary. If we assume that M is a manifold with non-empty boundary, the examples above of configuration spaces and pure mapping class groups have the extra structure of an FI#-module that allows us to conclude the following results from the arguments in Section 6.3.

Theorem 1.6. *Let $\Sigma = \Sigma_{g,r}$ be a connected compact oriented surface with non-empty boundary ($r > 0$). For any $i \geq 0$ and $n \geq 0$, each of the following invariants of $\text{PMod}_{g,r}^n$ is given by a polynomial in n of degree at most $2i$:*

- The i^{th} rational Betti number $b_i(\text{PMod}_{g,r}^n)$ and the i^{th} mod- p Betti number of $\text{PMod}_{g,r}^n$.
- The rank of $H^i(\text{PMod}_{g,r}^n; \mathbb{Z})$ and the rank of the p -torsion part of $H^i(\text{PMod}_{g,r}^n; \mathbb{Z})$.

Theorem 1.7. *Let M be a manifold with non-empty boundary that satisfies the hypothesis of Theorem 1.3. For $n \geq 0$ each of the following is given by a polynomial in n :*

- The i^{th} rational Betti number $b_i(\text{PMod}^n(M))$ and the i^{th} mod- p Betti number of $\text{PMod}^n(M)$.
- The rank of $H^i(\text{PMod}^n(M); \mathbb{Z})$ and the rank of the p -torsion part of $H^i(\text{PMod}^n(M); \mathbb{Z})$.

The polynomial is of degree at most i for rational Betti numbers and degree at most $2i$ in the other cases.

Closed Surfaces. For a fixed $n \geq 0$, we can relate the mapping class group of a closed surface with the one of a surface with non-empty boundary. Let

$$\delta_g : \text{PMod}_{g,1}^n \rightarrow \text{PMod}_g^n$$

be the group homomorphism induced by gluing a disk to the boundary component. And, whenever $r > 0$, let

$$\beta_g : \text{PMod}_{g,r}^n \rightarrow \text{PMod}_{g,r+1}^n$$

be the group homomorphism induced by gluing a pair of pants to one of the boundary components.

The following result is part of the so called Harer's stability Theorem and was proved initially by Harer ([Har85]). A proof of it with the improved bounds that we use can be found in [Wah].

Theorem 1.8 (Harer). *If $i \leq \frac{2}{3}g$, we have following isomorphisms:*

- (a) $H_i(\beta_g) : H_i(\text{PMod}_{g,r}^n; \mathbb{Z}) \rightarrow H_i(\text{PMod}_{g,r+1}^n; \mathbb{Z})$, if $r > 0$.
- (b) $H_i(\delta_g) : H_i(\text{PMod}_{g,1}^n; \mathbb{Z}) \rightarrow H_i(\text{PMod}_g^n; \mathbb{Z})$.

When the genus of the surface is large, by combining the previous result with Theorem 1.6 we obtain the following information in the case of closed surfaces.

Theorem 1.9. *If $g \geq \max\{2, \frac{3}{2}i\}$, then each of the following invariants of PMod_g^n is given by a polynomial in n for $n \geq 0$:*

- the i -th rational Betti number $b_i(\text{PMod}_g^n)$
- the rank of $H^i(\text{PMod}_g^n; \mathbb{Z})$
- the rank of the p -torsion part of $H^i(\text{PMod}_g^n; \mathbb{Z})$

In each case the polynomial is of degree at most $2i$.

1.3 Cohomological stability of some wreath products

Notation: The *surface pure braid group* is the group $\pi_1(\text{Conf}_n(\Sigma_{g,r}))$ and will be denoted by $P_n(\Sigma_{g,r})$. The *surface braid group* is $\pi_1(\text{Conf}_n(\Sigma_{g,r})/S_n)$ and we use $B_n(\Sigma_{g,r})$ to denote it. When $g = 0$ and $r = 1$, these are the pure braid group P_n and the braid group B_n , respectively. On the other hand, the *braid permutation group* Σ_n^+ is the group of string motions that preserve orientation of the circles (see [Wil12, Section 8] for a precise definition).

In Section 7 we discuss how our previous results and the closure of finite generation of FI-modules under tensor products can be used to get information about homological stability of some wreath products.

Theorem 1.10. *Let G be any group of type FP_∞ and let K_n be one of the following groups:*

- (i) *The symmetric group S_n ,*
- (ii) *The surface braid group $B_n(\Sigma_{g,r})$, with $g, r \geq 0$,*
- (iii) *The mapping class group $\text{Mod}_{g,r}^n$, with $2g + r > 2$,*
- (iv) *The mapping class group $\text{Mod}^n(M)$, where M is a smooth connected manifold of dimension $d \geq 3$ such that the hypotheses in Theorem 1.3 are satisfied,*
- (v) *The braid permutation group Σ_n^+ .*

Then for any $i \geq 0$ and any partition λ we have that

$$H^i(G \wr K_n; V(\lambda)_n) \approx H^i(G \wr K_{n+1}; V(\lambda)_n),$$

for any n sufficiently large.

Remark: The wreath product $G \wr K_n$ above is the semidirect product $G^n \rtimes K_n$, where K_n acts on G^n through the surjection $K_n \rightarrow S_n$.

In particular, plugging in the trivial representation for $V(\lambda)_n$, we obtain the following.

Corollary 1.11. *Let K_n be any of the groups above and let G be a group of type FP_∞ . Then the wreath product $G \wr K_n$ satisfies rational homological stability.*

Remarks: In general we do not have explicit stable ranges. The following is known about stable ranges:

- For (i) we get the stable range $n \geq 2i$. Homological stability is known to hold integrally in this case for $n \geq 2i + 1$ (see [HW10, Propositions 1.6]). Therefore our bound suggests that the possible failure of injectivity when $n = 2i$ should come from torsion.
- For the case (ii), Hatcher–Wahl have shown that if $r > 0$ the group $G \wr B_n(\Sigma_{g,r})$ satisfies integral homological stability when $n \geq 2i + 1$ ([HW10, Propositions 1.7]). Rationally, the stable range has been improved to $n \geq 2i$ by Randall-Williams (see [RW, Theorem A]).

1.4 Speculation on the existence of non-tautological classes in $\mathcal{M}_{g,n}$

The *tautological ring* of $\mathcal{M}_{g,n}$ is defined to be a subring $\mathcal{RH}^*(\mathcal{M}_{g,n})$ of the cohomology ring $H^*(\mathcal{M}_{g,n}; \mathbb{Q})$ generated by certain “geometric classes”: the kappa-classes $\kappa_j \in H^{2j}(\mathcal{M}_{g,n}; \mathbb{Q})$, for $j \geq 0$, and the psi-classes $\psi_i \in H^2(\mathcal{M}_{g,n}; \mathbb{Q})$, for $1 \leq i \leq n$. In $\mathcal{RH}^*(\mathcal{M}_{g,n})$, the class κ_j has grading j and ψ_i has grading 1 (half the cohomological grading). We refer the reader to [FP, Section 1] for precise definitions of the tautological rings of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$.

In [CEF, Section 5.1] it is proved that $\mathcal{RH}^*(\mathcal{M}_{g,\bullet})$ is a graded FI-module of finite type for $g \geq 2$. This follows from the fact that this graded FI-module is a quotient of the free commutative algebra

$$\mathbb{Q}[\{\kappa_j : j \geq 0\} \cup \{\psi_i : 1 \leq i \leq n\}],$$

where S_n acts trivially on the kappa-classes and permutes the psi-classes. From this description, we can see that for any $k \geq 0$ the weight of the FI-module $\mathcal{RH}^k(\mathcal{M}_{g,\bullet})$ is at most k . As a consequence we obtain an upper bound for the length of the representation $\ell(\mathcal{RH}^k(\mathcal{M}_{g,n})) \leq k + 1$.

On the other hand, Faber and Pandharipande studied in [FP] the S_n -action on $H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ and get an upper bound for the length of the irreducible representations occurring in the tautological ring $\mathcal{RH}^*(\overline{\mathcal{M}}_{g,n})$. Their interest is to exhibit, by other methods (counting, boundary geometry), several classes of Hodge type that cannot be tautological classes because the lengths of the corresponding S_n -representations are larger than their upper bound. In particular, they are almost positive to have established the existence of many non-tautological cohomology classes of Tate type on $\overline{\mathcal{M}}_{2,21}$. They obtained that $\ell(\mathcal{RH}^k(\overline{\mathcal{M}}_{g,n})) \leq k + 1$ ([FP, Section 4]). Since $\mathcal{RH}^k(\overline{\mathcal{M}}_{g,n})$ surjects onto $\mathcal{RH}^k(\mathcal{M}_{g,n})$, that implies that $\ell(\mathcal{RH}^k(\mathcal{M}_{g,n})) \leq k + 1$, which is the same bound that we obtained directly with the FI-module approach. In contrast, their method involves studying representations induced from the boundary strata.

Finally we would like to point out that from Table 1 we have the upper bounds

$$\ell(H^{2k}(\mathcal{M}_{g,n}; \mathbb{Q})) \leq 4k + 1.$$

This, contrasted with $\ell(\mathcal{RH}^k(\mathcal{M}_{g,n})) \leq k + 1$, suggests that there is room for the existence of non-tautological classes $\mathcal{M}_{g,n}$ and that an approach à la Faber and Pandharipande could demonstrate that some explicit classes are non-tautological. However, we have no indication that our bounds are sharp. As matter of fact, the only completely known case show evidence of the contrary since $H^2(\mathcal{M}_{g,n}; \mathbb{Q}) = \mathcal{RH}^1(\mathcal{M}_{g,n})$ has length 2.

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2 Preliminaries

In this section we summarize notions introduced in [CEF] and [CF] and set notation that will be recurrent in the rest of the paper. We refer the interested reader to [CEF, Sections 1 & 2] for precise results and proofs.

2.1 FI and FI#-modules

Let **FI** be the category whose objects are natural numbers \mathbf{n} and the morphisms $\mathbf{m} \rightarrow \mathbf{n}$ are injections from $\{1, \dots, m\}$ to $\{1, \dots, n\}$. Similarly we denote by **FI#** the category whose objects are natural numbers \mathbf{n} and the morphisms $\mathbf{m} \mapsto \mathbf{n}$ are triples (A, B, ψ) , where $A \subset \{1, \dots, m\}$, $B \subset \{1, \dots, n\}$ and $\psi : A \rightarrow B$ is a bijection.

Definition: An *FI-module* over a commutative ring k is a functor V from the category **FI** to the category \mathbf{Mod}_k of modules over k . An *FI#-module* over k is a functor V from **FI#** to \mathbf{Mod}_k . We denote $V(\mathbf{n})$ by V_n and $V(f)$ by f_* , for any $f \in \text{Hom}_{\mathbf{FI}}(\mathbf{m}, \mathbf{n})$. In the same manner a functor $V = \bigoplus V^i$ from **FI** to the category graded modules over k is called a *graded FI-module* over k . In particular, each V^i is an FI-module over k .

The category $\mathbf{FI-Mod}_k$ of FI-modules over k is an abelian category. The concepts of kernel, cokernel, sub-FI-module, quotient, injection and surjection are defined “pointwise” (for each $n \geq 1$). For us k will be either a field or \mathbb{Z} . Most of the examples that we consider are finite dimensional vector spaces over \mathbb{Q} , unless otherwise specified. Therefore, we use the notation **FI-Mod** for the category of FI-modules over \mathbb{Q} .

In the category $\mathbf{FI-Mod}_k$ we can define analogous concepts of the basic definitions coming from module theory.

Definition: An FI-module V over k is said to be *finitely generated in degree $\leq m$* if there exist v_1, \dots, v_s , with each $v_i \in V_{n_i}$ and $n_i \leq m$, such that V is the minimal sub-FI-module of V containing v_1, \dots, v_s . We write $V = \text{span}(v_1, \dots, v_s)$. An FI#-module over k is *finitely generated in degree $\leq m$* if the underlying FI-module is finitely generated in degree $\leq m$. A graded FI-module V over k is said to be of *finite type* if each FI-module V^i is finitely generated.

Finitely generated FI-modules have strong closure properties, in particular extensions and quotients of finitely generated FI-modules are still finitely generated ([CEF, Proposition 2.17]). In some of our examples below, the FI-module V over k has actually the extra structure of an FI#-module over k . In that case more is true: if V is finitely generated in degree $\leq m$, then any sub-FI#-module is finitely generated in degree $\leq m$ (follows from [CEF, Corollaries 2.25 & 2.26]).

Notation (The FI-modules $M(W)$): Let $m \in \mathbb{N}$ and consider a fixed S_m -representation W over a field k or $k = \mathbb{Z}$. The FI-module $M(W)$ is defined as follows:

$$M(W)_n := \begin{cases} 0, & \text{if } n < m \\ \text{Ind}_{S_m \times S_{n-m}}^{S_n} W \boxtimes k, & \text{if } n \geq m. \end{cases}$$

In particular, when $W = k[S_m]$ we will denote the FI-module $M(W)$ by $M(m)$. These FI-modules were introduced in [CEF, Section 2.1]. By definition, they are finitely generated in degree m . Moreover they have the structure of an FI#-module and have surjectivity degree at most m .

The FI#-modules $M(W)$ are “building blocks” for general FI#-modules ([CEF, Theorem 2.24 and Corollary 2.26]).

2.2 FI-modules over fields of characteristic zero

If k is a field of characteristic zero, finite generation of FI-modules is also closed under taking submodules (“Noetherian property” [CEF, Theorem 2.60]). The proof of the “Noetherian property” does not give an upper bound on the degree in which a given subobject is generated. To deal with this the notion of *weight of an FI-module* was introduced in [CEF, Section 2.5].

Definition: The collection of FI-modules over k of *weight $\leq d$* is the minimal collection which contains all FI-modules generated in degree $\leq d$ and is closed under subquotients and extensions. An FI-module V is of *finite weight* if it is of weight $\leq d$ for some $d \geq 0$. We denote by $\text{weight}(V)$ the minimum of such d (as pointed out in [CEF, Remark 2.55] this is an alternate definition to [CEF, Definition 2.50]).

The subgroup of S_n that permutes $\{a+1, \dots, n\}$ and acts trivially on $\{1, 2, \dots, a\}$ is denoted by S_{n-a} . The coinvariant quotient $(V_n)_{S_{n-a}}$ is the S_a -module $V_n \otimes_{k[S_{n-a}]} k$, i.e. the largest quotient of V_n on which S_{n-a} acts trivially.

The following provides a notion of stabilization and range of stabilization for an FI-module (this is just a rephrasing of [CEF, Definitions 2.34 & 2.35]).

Definition: Let V be an FI-module over k . If for every $a \geq 0$ and $n \geq N+a$ the map of coinvariants

$$(V_n)_{S_{n-a}} \rightarrow (V_{n+1})_{S_{n-a}} \quad (3)$$

induced by the standard inclusion $I_n : \{1, \dots, n\} \hookrightarrow \{1, \dots, n, n+1\}$, is an injection of S_a -modules, we say that V has *injectivity degree* $\leq N$. If the map (3) is surjective, we say that V has *surjectivity degree* $\leq N$. The FI-module V has *stability type* (M, N) if it has injectivity degree M and surjectivity degree N . The *stability degree* of V is given by at most $\max(M, N)$. When V is an $FI\#$ -module, the identity on V_n factors through I_n , hence the injectivity degree is always 0.

2.3 Relation with representation stability

An FI-module over a field k of characteristic zero encodes the information of a sequence of S_n -representations. More is true. The following definition was introduced in [CF]:

Definition: A sequence $\{V_n\}_{n=1}^\infty$ of finite dimensional S_n -representations over k with linear maps $\phi_n : V_n \rightarrow V_{n+1}$ is said to be *consistent* if the maps $\phi_n : V_n \rightarrow V_{n+1}$ are equivariant with respect to the natural inclusion $S_n \hookrightarrow S_{n+1}$.

If V is an FI-module and I_n the standard inclusion, then $\{V_n, V(I_n)\}$ is a consistent sequence of S_n -representations. We are interested in the behavior of consistent sequences arising from FI-modules.

Notation (Representations of S_n in characteristic zero): The irreducible representations of S_n over a field of characteristic zero k are classified by partitions λ of n . By a partition of n we mean $\lambda = (\lambda_1 \geq \dots \geq \lambda_l > 0)$ where $l \in \mathbb{Z}$ and $\lambda_1 + \dots + \lambda_l = n$. We will write $|\lambda| = n$. The corresponding irreducible S_n -representation will be denoted by V_λ . Every V_λ is defined over \mathbb{Q} and any S_n -representation decomposes over \mathbb{Q} into a direct sum of irreducibles ([FH91] is a standard reference). The decomposition of an S_n -representation over any such field k does not depend on k .

If λ is any partition of m , i.e. $|\lambda| = m$, then for any $n \geq |\lambda| + \lambda_1$ the *padded partition* $\lambda[n]$ of n is given by $\lambda[n] = (n - |\lambda|, \lambda_1, \dots, \lambda_l)$. Keeping the notation from [CF] we set $V(\lambda)_n = V_{\lambda[n]}$ for any $n \geq |\lambda| + \lambda_1$. Every irreducible S_n -representation is of the form $V(\lambda)_n$ for a unique partition λ . For a given partition $|\lambda| = m$, we use $M(\lambda)$ to denote the FI-module $M(V_\lambda)$.

We define the *length* of an irreducible representation of S_n to be the number of parts in the corresponding partition of n . The trivial representation has length 1, and the alternating representation has length n . We define the *length* $\ell(V)$ of a finite dimensional representation V of S_n to be the maximum of the lengths of the irreducible constituents. Notice that $\ell(V_\lambda) \leq |\lambda|$.

The notion of representation stability of sequences of rational S_n -representations was first defined in [CF]. We recall this definition here.

Definition: A consistent sequence $\{V_n, \phi_n\}_{n=1}^\infty$ of finite dimensional S_n -representations is said to be *uniformly representation stable with stable range* $n \geq N$ if the following conditions are satisfied for all $n \geq N$:

- I. **Injectivity.** The maps $\phi_n: V_n \rightarrow V_{n+1}$ are injective.
- II. **Surjectivity.** The S_{n+1} -span of $\phi_n(V_n)$ equals V_{n+1} .
- III. **Uniformly Multiplicity Stable.** For each partition λ , the multiplicities $c_\lambda(V_n)$ of $V(\lambda)_n$ in V_n are constant in n .

The original motivation to introduce the notion of an FI-module was to give a new approach to representation stability that would allow to simplify many of the arguments. The correspondence between finitely generated FI-modules and uniformly representation stable sequences is the content of [CEF, Section 2.6].

3 A spectral sequence argument

In this section we present the general spectral sequence argument that will give us finite generation and specific bounds in Theorems 1.1 and 5.3. We basically apply the idea used in the proof of [CEF, Theorem 4.2] to a more general context.

Setting: Suppose that we have a first quadrant spectral sequence of FI-modules $E_*^{p,q}$ over \mathbb{Q} converging to a graded FI-module $H^*(E)$ over \mathbb{Q} . Let α and β be two non-negative constants such that $2\alpha \leq \beta$. In what follows, we assume that for any $p, q \geq 0$ the FI-module $E_2^{p,q}$ is finitely generated with injectivity degree at most βq and surjectivity degree at most $\alpha p + \beta q$.

In our applications below, $E_*^{p,q}$ is either a Leray, Leray–Serre or Hochschild–Serre spectral sequence.

Lemma 3.1. *For any $p, q \geq 0$ and $r \geq 3$, the FI-module $E_r^{p,q}$ is finitely generated with injectivity degree at most $\alpha p + \beta q + (\beta - \alpha)r + (\alpha - 2\beta)$ and surjectivity degree at most $\alpha p + \beta q$.*

Proof. Finite generation of an FI-module is closed under subquotients. To verify the stated stability type we proceed by induction on $r \geq 3$. The base of induction is the case $r = 3$. To compute $E_3^{p,q}$ we consider the complex of FI-modules

$$E_2^{p-2,q+1} \longrightarrow E_2^{p,q} \longrightarrow E_2^{p+2,q-1},$$

where the left map is the differential $d_2^{p-2,q+1}$ and the right map is $d_2^{p,q}$. By hypothesis the left hand side term in the previous complex has surjectivity degree at most $\alpha(p-2) + \beta(q+1) = \alpha p + \beta q + (\beta - 2\alpha)$. The middle term has stability type at most $(\beta q, \alpha p + \beta q)$ and the right hand side term has injectivity degree at most $\beta(q-1)$. Hence, by applying [CEF, Proposition 2.45] to the complex of FI-modules above, we obtain that the quotient FI-module

$$E_3^{p,q} \approx \ker d_2^{p,q} / \operatorname{im} d_2^{p-2,q+1}$$

has injectivity degree at most

$$\max(\alpha p + \beta q + (\beta - 2\alpha), \beta q) = \alpha p + \beta q + (\beta - 2\alpha) = \alpha p + \beta q + (\beta - \alpha)(3) + (\alpha - 2\beta)$$

since $2\alpha \leq \beta$, and surjectivity degree at most

$$\max(\alpha p + \beta q, \beta q - \beta) = \alpha p + \beta q$$

since $\alpha, \beta \geq 0$.

Now suppose that the statement is true for $E_r^{p,q}$. To compute $E_{r+1}^{p,q}$ we consider the complex of FI-modules

$$E_r^{p-r, q+r-1} \longrightarrow E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1},$$

where the left map is the differential $d_r^{p-r, q+r-1}$ and the right map $d_r^{p,q}$. By induction, the left hand side term in the previous complex has surjectivity degree at most $\alpha p + \beta q + (\beta - \alpha)(r+1) + (\alpha - 2\beta)$. The middle term has stability type at most

$$(\alpha p + \beta q + (\beta - \alpha)r + (\alpha - 2\beta), \alpha p + \beta q).$$

Finally the right hand side term has injectivity degree at most $\alpha p + \beta q + \alpha - \beta$. By applying again [CEF, Proposition 2.45] we get the desired stability type for the quotient

$$E_{r+1}^{p,q} \approx \ker d_r^{p,q} / \operatorname{im} d_r^{p-r, q+r-1}.$$

■

For a given $i \geq 0$ and $0 \leq p \leq i$, we have that $E_\infty^{p, i-p}(n) = E_{i+2}^{p, i-p}(n)$. From Lemma 3.1 we get the immediate corollary.

Corollary 3.2. *The FI-module $E_\infty^{p, i-p}$ has injectivity degree at most*

$$\alpha p + \beta(i-p) + (\beta - \alpha)(i+2) + (\alpha - 2\beta) = (2\beta - \alpha)i + (\alpha - \beta)p - \alpha \leq (2\beta - \alpha)i - \alpha$$

and surjectivity degree at most

$$\alpha p + \beta(i-p) = \beta i + (\alpha - \beta)p \leq \beta i.$$

As assumed at the beginning of this section, the spectral sequence $E_*^{p,q}$ converges to a graded FI-module $H^*(E)$. From Lemma 3.1 and Corollary 3.2 we can conclude the following about the stability type of each FI-module $H^i(E)$.

Theorem 3.3. *For any $i \geq 0$ the FI-module $H^i(E)$ is finitely generated with stability type at most $((2\beta - \alpha)i - \alpha, \beta i)$.*

Proof. For each $i \geq 0$, there is a natural filtration of $H^i(E)$ by FI-modules

$$0 \subseteq F_i^i \subseteq F_{i-1}^i \subseteq \dots \subseteq F_1^i \subseteq F_0^i = H^i(E), \quad (4)$$

where, for $0 \leq p \leq i$, the successive quotients $F_p^i / F_{p+1}^i \approx E_\infty^{p, i-p}$. The theorem follows from combining the bounds in Lemma 3.2 with [CEF, Proposition 2.46], which states injectivity and surjectivity degrees for filtrations of FI-modules satisfying the conditions above. ■

Remark: In the examples below we know the weight of each FI-module $E_2^{p,q}$ and we use it to obtain specific bounds for the weight of the FI-modules $H^i(E)$.

3.1 Spectral sequences and FI#-modules

We conclude this section with an argument that allows us to take advantage of the extra structure of finitely generated FI#-modules to get information about the cases where k is a field of arbitrary characteristic or \mathbb{Z} . This follows essentially the proof of [CEF, Theorem 4.7].

Setting: Suppose that we have a first quadrant spectral sequence of FI-modules $E_*^{p,q}$ over k converging to a graded FI#-module $H^*(E; k)$ over k . Let α and β be two non-negative constants such that $\alpha \leq \beta$. Assume that for any $p, q \geq 0$ each term $E_2^{p,q}$ is an FI#-module which is finitely generated in degree $\leq \alpha p + \beta q$.

Theorem 3.4. *Let k be any field or \mathbb{Z} . For any $i \geq 0$ the FI#-module $H^i(E; k)$ is finitely generated in degree $\leq \beta i$.*

Proof. Suppose first that k is a field. We have that $E_2^{p,q}$ is an FI#-module which is finitely generated in degree $\leq \alpha p + \beta q$. [CEF, Corollary 2.27] allows to relate this upper bound on the degree of generation with the dimension of the k -vector space $E_2^{p,q}(n)$ and conclude that $\dim_k E_2^{p,q}(n) = O(n^{\alpha p + \beta q})$. Since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$ and k is a field, then $\dim_k E_\infty^{p,q}(n) = O(n^{\alpha p + \beta q})$. Finally for each $i \geq 0$, from the filtration (4) of $H^i(E; k)$, we have that $\dim_k H^i(E; k) = O(n^{\beta i})$ (since $\alpha p + \beta(i - p) \leq \beta i$ for any $0 \leq p \leq i$). Hence, by applying again [CEF, Corollary 2.27] we get the desired implication. The case when $k = \mathbb{Z}$ can be treated similarly because the rank of a \mathbb{Z} -module is non-increasing when passing to submodules. ■

4 Sequences of cohomology groups as FI-modules (part I)

In this section we focus on our first two sequences of cohomology groups (examples (1) and (2) in the introduction): $H^i(M^n; \mathbb{Q})$ and $H^i(\text{Conf}_n(M); \mathbb{Q})$. These examples have been previously understood as finitely generated FI-modules. Nevertheless we revisit them since they are key ingredients to understand our other examples in Section 6.

4.1 The FI-module $H^i(M^\bullet)$

Let M be a connected CW-complex with $\dim_{\mathbb{Q}}(H^i(M; \mathbb{Q})) < \infty$ for any $i \geq 0$. Consider the co-FI-space M^\bullet given by $\mathbf{n} \mapsto M^n$. If $f \in \text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{n})$, then the morphism $f^* : M^n \rightarrow M^m$ is defined by $f^*(x_1, \dots, x_n) = (x_{f(1)}, \dots, x_{f(m)})$. For each $i \geq 0$ we compose with the contravariant functor $H^i(-; \mathbb{Q})$ to get an FI-module.

Proposition 4.1. *For any $i \geq 0$, $H^i(M^\bullet)$ is an FI#-module finitely generated of weight $\leq i$ and has stability type at most $(0, i)$.*

Proof. This is a consequence of the Künneth formula. As pointed out in [CEF, Section 4] the graded FI-module $H^*(M^\bullet)$ almost coincides with the FI-module $H^*(M)^{\otimes \bullet}$ (see [CEF, Definition 2.71]). It can actually be shown that the FI-module $H^i(M^\bullet)$ is a direct sum of FI-modules of the form $M(W_k)$, where W_k is some S_k -representation and each summand satisfies that $k \leq i$ (see for example [JR11, Proposition 6.5]). Then the weight and the stability type claimed in Proposition 4.1 follow. ■

If G is a group of type FP_∞ , we can consider the co-FI-group G^\bullet . If $M = K(G, 1)$, then the FI-module $H^i(G^\bullet) = H^i(M^\bullet)$ and Proposition 4.1 applies as well.

4.2 Cohomology of configuration spaces

Let M be a connected, oriented manifold of dimension $d \geq 2$ with $\dim_{\mathbb{Q}}(H^*(M; \mathbb{Q})) < \infty$ and let $\text{Conf}_\bullet(M)$ be the functor that assigns $\mathbf{n} \mapsto \text{Conf}_n(M)$ and with morphisms defined as in the case of M^\bullet . Since the inclusion $\text{Conf}_n(M) \hookrightarrow M^n$ is S_n -equivariant, we get a corresponding map of co-FI-spaces $\text{Conf}_\bullet(M) \rightarrow M^\bullet$. We recall here how a spectral sequence argument can be used to obtain finiteness conditions for the FI-modules $H^q(\text{Conf}_\bullet(M))$.

Let us take, together for all n , the Leray spectral sequences of $\text{Conf}_n(M) \hookrightarrow M^n$. The functoriality of the Leray spectral sequence implies that we have a spectral sequence of FI-modules

$$E_*^{p,q} = E_*^{p,q}(\text{Conf}_\bullet(M) \rightarrow M^\bullet)$$

converging to the graded FI-module $H^*(\text{Conf}_\bullet(M))$.

Using this spectral sequence, finite type of this graded FI-module has been proved for $d \geq 2$ and particular bounds for the stability degree have been obtained for the case when dimension of M is $d \geq 3$.

Theorem 4.2 ([CEF], Theorem 4.2). *Let M be a connected, oriented manifold of dimension $d \geq 3$. For any $i \geq 0$, the FI-module $H^i(\text{Conf}_\bullet(M))$ has weight $\leq i$ and stability type at most $(i + 2 - d, i)$.*

We now focus in the case where Σ is a connected, oriented surface ($d = 2$).

Following the approach in [CEF, Section 4] we get a better bound for the degree of the FI-module $H^i(\text{Conf}_\bullet(\Sigma))$ and get the specific bounds for the stability type indicated in Theorem 1.1.

Proof. (of Theorem 1.1) We have a spectral sequence of FI-modules

$$E_*^{p,q} = E_*^{p,q}(\text{Conf}_\bullet(\Sigma) \rightarrow \Sigma^\bullet)$$

converging to the graded FI-module $H^*(\text{Conf}_\bullet(\Sigma))$. For any $p, q \geq 0$ the FI-module $E_2^{p,q}$ is the direct sum of FI-modules of the form $M(W_k)$ where W_k is certain S_k -representation. Moreover, each summand satisfies $k \leq p + 2q$ (see [Chu, Section 3.3.]). Hence, for every $p, q \geq 0$ we have that $E_2^{p,q}$ is finitely generated in degree $\leq p + 2q$ and has stability type at most $(0, p + 2q)$. This is precisely the setting needed for our spectral sequence argument in Section 3 with constants $\alpha = 1$ and $\beta = 2$. Then for each $i \geq 0$ the FI-module $H^i(\text{Conf}_\bullet(\Sigma))$ is finitely generated with stability type at most $(3i - 1, 2i)$.

In addition, Totaro proved in [Tot96, Theorem 3] that if M is a smooth complex projective variety, then $E_\infty(\text{Conf}_n(M) \hookrightarrow M^n) = E_3$. This is the case when Σ is a closed surface. Therefore we can use Lemma 5.2 to improve the bounds for the stability type of $E_\infty^{p,i-p}$ to be at most $(2i, 2i)$, which gives the corresponding bounds stated before.

On the other hand, if $\partial\Sigma$ is nonempty, [CEF, Proposition 4.6] implies that $H^i(\text{Conf}_\bullet(\Sigma))$ has an FI#-module structure and the injectivity degree is 0.

Finally, observe that for any $0 \leq i$ and $0 \leq p \leq i$ the FI-module $E_\infty^{p,i-p}$ is a subquotient of the FI-module $E_2^{p,i-p}$ of weight $p + 2q$. It follows that $\text{weight}(E_\infty^{p,i-p}) \leq 2i - p \leq 2i$, which implies that $H^i(\text{Conf}_\bullet(\Sigma))$ has weight at most $2i$. ■

Remark: In [CEF, Section 2.6] finite generation of an FI-module is related with representation stability. In particular, Theorem 1.1 together with [CEF, Proposition 2.58] imply that the sequence $H^i(\text{Conf}_n(\Sigma); \mathbb{Q})$ is uniformly representation stable and we recover the stable range of $n \geq 4i$ for the cases when Σ is closed or has non-empty boundary, which was first obtained in [Chu, Theorem 1].

Finally for manifolds with non-empty boundary it turns out that $H^i(\text{Conf}_\bullet(M); k)$ is an FI#-module for any commutative ring k . The argument of Section 3.1 implies the following result.

Theorem 4.3 (Theorem 4.7 in [CEF]). *Let M be a connected, oriented manifold of dimension $d \geq 2$ which is the interior of a compact manifold with non-empty boundary. Let k be any field or \mathbb{Z} , then for each $i \geq 0$ the FI#-module $H^i(\text{Conf}_\bullet(M); k)$ over k is finitely generated by $O(n^{2i})$ elements.*

5 FI[G]-modules

Here we introduce the notion on an FI[G]-module. Basically we want to incorporate the action of a group G on our sequences of S_n -representations. These types of FI-modules will allow us to construct new FI-modules by taking cohomology with twisted coefficients. We will see how in some situations we can use finite generation of the original FI[G]-module to get finite generation and specific bounds for the new FI-modules. In Section 5.2 we use this setting in spectral sequence arguments for cohomology of fibrations and groups extensions.

Definition: Let k be any commutative ring and let G be a group. An FI[G]-module V over k is a functor from the category **FI** to the category **G-Mod** $_k$ of G -modules over k . We say that an FI[G]-module V is *finitely generated* if it is finitely generated as an FI-module. Similarly an FI#[G]-module V over k is a functor from the category **FI#** to the category **G-Mod** $_k$.

FI[G]-modules and consistent sequences compatibles with G -actions: For an FI[G]-module V , for each $\sigma \in S_n$ the induced linear automorphism $\sigma_* : V_n \rightarrow V_n$ is a G -map. Hence the S_n -action and the G -action on V_n commute. If we denote by ϕ_n the map obtained by applying V the standard inclusion I_n (i.e. $\phi_n = V(I_n)$), we have that $\{V_n, \phi_n\}$ is a *consistent sequence of S_n -representations compatible with G -actions* in the sense of [JR11].

5.1 Getting new FI-modules from FI[G]-modules

Let V be an FI[G]-module. Consider a path connected space X with fundamental group G . For each integer $p \geq 0$ we have a covariant functor $H^p(X; _)$ from the category **G-Mod** to itself. Hence we have a new FI[G]-module $H^p(X; V)$ where $H^p(X; V)_n := H^p(X; V_n)$, the p th cohomology of X with local coefficients in the G -module V_n (see [Hat02, Section 3.H]). Moreover the functor $H^*(X; V)$ given by $H(X; V)_n := H^*(X; V_n)$ is a graded FI-module. We are now ready to present the proof of Theorem 1.5.

Proof. (of Theorem 1.5) Given that $G = \pi_1(X)$, the universal cover \tilde{X} of X has a G -equivariant cellular chain complex. Since X has finitely many cells in each dimension, for each $p \geq 0$ the group $C_p(\tilde{X})$ is a free G -module of finite rank, say $C_p(\tilde{X}) \approx (\mathbb{Z}G)^{d_p}$. A preferred G -basis x_1, \dots, x_{d_p} can be provided by selecting a p -cell in \tilde{X} over each cell in X .

For each $p \geq 0$ and $n \in \mathbb{N}$ we have an isomorphism of G -modules $\mathcal{H}om_G(C_p(\tilde{X}), V_n) \approx V_n^{\oplus d_p}$, given by $h \mapsto (h(x_1), \dots, h(x_{d_p}))$. Moreover, for any morphism $\phi : V_m \rightarrow V_n$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}om_G(C_p(\tilde{X}), V_m) & \xrightarrow{\phi \circ _} & \mathcal{H}om_G(C_p(\tilde{X}), V_n) \\ \approx \downarrow & & \downarrow \approx \\ V_m^{\oplus d_p} & \xrightarrow{\phi^{\oplus d_p}} & V_n^{\oplus d_p} \end{array}$$

Therefore the FI[G]-module $C^p(X; V)$ given by $C^p(X; V)_n := \mathcal{H}om_G(C_p(\tilde{X}), V_n)$ is precisely the direct sum of FI[G]-modules $V^{\oplus d_p}$.

Since the weight of an FI-module does not increase when taking extensions, we have that $V^{\oplus d_p}$ is finitely generated of weight $\leq m$. Moreover, we can filter

$$V^{\oplus d_p} \supseteq V^{\oplus d_p - 1} \supseteq \dots \supseteq V \supseteq 0$$

and since V has stability degree N , [CEF, Proposition 2.46] implies that $V^{\oplus d_p}$ has stability degree N .

Furthermore, the $\mathbf{FI}[G]$ -module $H^p(X; V)$ is obtained from the complex of $\mathbf{FI}[G]$ -modules

$$C^{p-1}(X, V) \xrightarrow{\delta_{p-1}} C^p(X, V) \xrightarrow{\delta_p} C^{p+1}(X, V)$$

where we have that each $\mathbf{FI}[G]$ -module has stability degree N and is finitely generated of degree $\leq m$. The weight of an \mathbf{FI} -module is preserved under subquotients and from [CEF, Proposition 2.45] applied to the previous complex we get the desired stability degree. ■

Remark: For each integer $p \geq 0$ we have a covariant functor $H^p(G; _)$ from $\mathbf{G-Mod}$ to itself (see [Bro94]), hence we have the $\mathbf{FI}[G]$ -module $H^p(G; V)$ given by $H^p(G; V)_n := H^p(G; V_n)$. If G is a group of type FP_∞ , then the space $X = K(G, 1)$ satisfies the hypothesis of Theorem 1.5 and the $\mathbf{FI}[G]$ -module $H^p(X; V)$ is precisely $H^p(G; V)$.

The case of $\mathbf{FI}\#[G]$ -modules: Next we state the equivalent result to Theorem 1.5 when we take coefficients in a finitely generated $\mathbf{FI}\#[G]$ -module.

Theorem 5.1 (Cohomology with coefficients in a f.g. $\mathbf{FI}\#[G]$ -module). *Let k be any field or \mathbb{Z} and suppose that G is the fundamental group of a connected CW complex X with finitely many cells in each dimension. If V is an $\mathbf{FI}\#[G]$ -module over k finitely generated in degree $\leq m$, then for every $p \geq 0$, the $\mathbf{FI}\#$ -module $H^p(X; V)$ is finitely generated in degree $\leq m$.*

Proof. Clearly $H^p(X; V)$ is a covariant functor from $\mathbf{FI}\#$ to \mathbf{Mod}_k . First suppose that k is any field. Keeping the notation from the previous proof we have that

$$\dim_k C^p(X, V_n) = \dim_k V_n^{\oplus d_p} = O(n^m).$$

By hypothesis and [CEF, Corollary 2.27] it follows that $\dim_k V_n = O(n^m)$. Then dimension over k of the subquotient $H^p(X, V_n)$ is $O(n^m)$ and [CEF, Corollary 2.27] gives us the desired conclusion. A similar argument applies for $k = \mathbb{Z}$ considering rank instead of dimension. ■

5.2 $\mathbf{FI}[G]$ -modules and spectral sequences

Let X be a connected CW complex with finitely many cells in each dimension and let $x \in X$ be a fixed base point. Suppose that the fundamental group $\pi_1(X, x)$ is G . Consider a functor from \mathbf{FI}^{op} to the category $\mathbf{Fib}(X)$ of fibrations over X (a *co-FI-fibration over X*). Let

$$E_n \rightarrow X$$

be the fibration associated to \mathbf{n} , and H_n the fiber over the basepoint x . We denote by E the co-FI space of total spaces $\mathbf{n} \mapsto E_n$ and by H the co-FI space of fibers $\mathbf{n} \mapsto H_n$. We can think of $E \rightarrow X$ as a pointwise fibration over X with “fiber” H .

Let us take, together for all n , the Leray-Serre spectral sequences associated to each fibration $E_n \rightarrow X$. The functoriality of the Leray-Serre spectral sequence implies that we have a spectral sequence of \mathbf{FI} -modules

$$E_*^{p,q} = E_*^{p,q}(E \rightarrow X)$$

converging to the graded \mathbf{FI} -module $H^*(E)$.

The E_2 -page of this spectral sequence is the $\mathbf{FI}[G]$ -module

$$E_2^{p,q} = H^p(X; H^q(H))$$

Remark: Observe that for any $q \geq 0$ and $n \geq 1$, we get an action of the fundamental group G on $H^q(H_n; \mathbb{Q})$ from the n -th fibration, which gives to the FI-module $H^q(H)$ the structure of an $\text{FI}[G]$ -module.

With this setting, we want to use Theorem 1.5 and the spectral argument given in Section 3 to determine finiteness conditions for the graded FI-module $H^*(E)$ given that we know that the $\text{FI}[G]$ -module $H^q(H)$ is finitely generated and we have upper bounds for the degree and the stability degree of this FI-module.

The typical situation that we will have in the examples in Section 6 below is that the $\text{FI}[G]$ -module $H^q(H)$ is finitely generated of weight $\leq \beta q$ with stability degree $\leq \beta q$, for some positive constant β . Then Theorem 1.5 gives us the following information about the E_2 -page.

Lemma 5.2. *Suppose that for any $q \geq 0$ the $\text{FI}[G]$ -module $H^q(H)$ is finitely generated of weight $\leq \beta q$ with stability degree $\leq \beta q$. Then, for any $p, q \geq 0$, the $\text{FI}[G]$ -module $E_2^{p,q} = H^p(X; H^q(H))$ has weight $\leq \beta q$ and stability degree $\leq \beta q$.*

For a given $i \geq 0$ and $0 \leq p \leq i$, the FI-module $E_\infty^{p,i-p}$ is a subquotient of $E_2^{p,i-p}$. Since the weight of an FI-module cannot increase when taking subquotients, it follows that the FI-module $E_\infty^{p,i-p}$ is finitely generated of weight $\leq \beta i$. Moreover, the spectral sequence gives a natural filtration of $H^i(E)$ by FI-modules

$$0 \subseteq F_i^i \subseteq F_{i-1}^i \subseteq \dots \subseteq F_1^i \subseteq F_0^i = H^i(E),$$

where, for $0 \leq p \leq i$, the FI-module F_p^i is an extension of F_{p+1}^i by $E_\infty^{p,i-p}$ of weight $\leq \beta i$. Since, by definition, the weight of an FI-module is preserved under extensions, therefore $H^i(E)$ has weight at most βi .

Furthermore, we have precisely the setting described in Section 3 for constants $\alpha = 0$ and $\beta > 0$ and Theorem 3.3 takes the following form.

Theorem 5.3. *For any $i \geq 0$ the FI-module $H^i(E)$ is finitely generated of weight at most βi and has stability type at most $(2\beta i, \beta i)$.*

The case of the group extensions: Let G be a group of type FP_∞ . Consider a functor from \mathbf{FI}^{op} to the category of group extensions with quotient G and isomorphisms of such (a *co-FI-group extension of G*). Let

$$1 \rightarrow H_n \rightarrow E_n \rightarrow G \rightarrow 1$$

be the group extension associated to \mathbf{n} and denote by E and H the corresponding co-FI groups $\mathbf{n} \mapsto E_n$ and by $\mathbf{n} \mapsto H_n$. For each group extension there is an associated fibration

$$K(E_n, 1) \rightarrow K(G, 1)$$

with fiber over a fixed base point $x \in K(G, 1)$ an Eilenberg-MacLane space $K(H_n, 1)$. Observe that the space $K(G, 1)$ has the homotopy type of a connected CW complex with finitely many cells in each dimension since G is of type FP_∞ . Hence, this gives us a functor from \mathbf{FI}^{op} to $\mathbf{Fib}(K(G, 1))$ as in the setting of Section 5.2 and we obtain the conclusion of Theorem 5.3 about the FI-modules $H^i(E)$.

Remarks: The Leray-Serre spectral sequence associated to the fibration above corresponds to the Hochschild-Serre spectral sequence associated to the original group extension. Notice that we could have considered this spectral sequence in our previous discussion.

The Hochschild-Serre spectral sequence and FI#-modules: Assume that we have a functor from $\mathbf{FI}\#^{op}$ to the category of group extensions with quotient G , and not just from \mathbf{FI}^{op} as before. By taking the Hochschild-Serre spectral sequence associated to each group extension with coefficients in any field k or \mathbb{Z} , we obtain with a first quadrant spectral sequence of $\mathbf{FI}\#$ -modules converging to the graded $\mathbf{FI}\#$ -modules $H^*(E; k)$. Furthermore, suppose that for any $q \geq 0$ the $\mathbf{FI}\#$ -module $H^q(H; k)$ is finitely generated over k in degree $\leq \beta q$, for some $\beta > 0$. Then Theorem 5.1 implies that for any $p, q \geq 0$, the $\mathbf{FI}\#$ -module $E_2^{p,q}$ is finitely generated in degree $\leq \beta q$. Since we have the setting from Section 3.1 with $\alpha = 0$ and $\beta > 0$, we can conclude that for any $i \geq 0$ the $\mathbf{FI}\#$ -module $H^i(E, k)$ is finitely generated in degree $\leq \beta i$.

6 Sequences of cohomology groups as FI-modules (part II)

Let us apply the perspective described in Section 5 to understand other sequences of cohomology groups as finitely generated FI-modules. Most of these sequences were already considered in [JR11]. We will see here how the FI-module approach allows us to obtain more information.

Now we are interested in the FI-modules that arise from the following examples of co-FI-spaces and co-FI-groups corresponding, respectively, to examples (3), (5) and (6) from the introduction:

- The co-FI-space $\mathcal{M}_{g,\bullet}$ given by $\mathbf{n} \mapsto \mathcal{M}_{g,n}$ and such that assigns for each $f \in \text{Hom}_{\mathbf{FI}}(\mathbf{m}, \mathbf{n})$, the map $f^* : \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,m}$ defined by $f^*([X; x_1, \dots, x_n]) = ([X; x_{f(1)}, \dots, x_{f(m)}])$.

Remark: Morphisms can be extended from $\mathcal{M}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$ so that the natural inclusions $\mathcal{M}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n}$ give us a map of co-FI-spaces $\mathcal{M}_{g,\bullet} \rightarrow \overline{\mathcal{M}}_{g,\bullet}$. See for example [KV07, Chapter 1] for a definition of $\overline{\mathcal{M}}_{g,n}$ and its corresponding forgetful maps.

- The co-FI-group $\text{PMod}^\bullet(M)$ which is the functor $\mathbf{n} \mapsto \text{PMod}^n(M)$. To define the morphisms, we denote by M^S the manifold M with $|S|$ points in the interior of M labeled by the elements in S a given finite subset of \mathbb{N} . Then, for any $f \in \text{Hom}_{\mathbf{FI}}(\mathbf{m}, \mathbf{n})$, we consider $f : M^{[n]} \rightarrow M^{f([m])}$ given by forgetting the points labeled by $[n] \setminus f([m])$. Here $[n]$ denotes the set $\{1, 2, \dots, n\}$. This induces the morphism $f_* : \text{PMod}^n(M) \rightarrow \text{PMod}^m(M)$, using the natural identification of $M^{f([m])}$ with $M^{[m]}$.
- The co-FI-space $B \text{Diff}^\bullet(M)$ given by $\mathbf{n} \mapsto B \text{Diff}^n(M)$ and where the morphisms are defined in a similar manner as for $\text{PMod}^\bullet(M)$.

6.1 Cohomology of moduli spaces $\mathcal{M}_{g,n}$ and pure mapping class groups of surfaces

Let $2g+r > 2$ and consider the functor from \mathbf{FI}^{op} to the category of group extensions of $G = \text{Mod}_{g,r}$ given as follows. The group extension associated to \mathbf{n} is

$$1 \rightarrow \pi_1(\text{Conf}_n(\Sigma_g^r)) \rightarrow \text{PMod}^n(\Sigma_{g,r}) \rightarrow \text{Mod}_{g,r} \rightarrow 1.$$

This is the *Birman exact sequence* introduced by Birman in [Bir74]. A proof of the exactness can be found in [FM12]. To see that this association is indeed functorial we refer the reader to [JR11, Section 5].

From Theorem 1.1 we have that the $\mathbf{FI}[G]$ -module $H^q(\pi_1[\text{Conf}_n(\Sigma_g^r)]) = H^q(\text{Conf}_n(\Sigma_g^r))$ is finitely generated of weight $\leq 2q$ with stability degree $\leq 2q$. From our discussion in Section 5.2 with $\beta = 2$, Theorem 1.2 follows.

6.2 Cohomology of pure mapping class groups for higher dimensional manifolds

Let M be a smooth connected manifold of dimension $d \geq 3$ and suppose that the fundamental group $\pi_1(M)$ has trivial center or $\text{Diff}(M)$ is simply connected. Moreover, we assume that $\text{Mod}(M)$ is of type FP_∞ .

Consider the functor from \mathbf{FI}^{op} that associates to each \mathbf{n} the group extensions of $G = \text{Mod}(M)$

$$1 \longrightarrow \pi_1(\text{Conf}_n(\overset{\circ}{M})) \longrightarrow \text{PMod}^n(M) \longrightarrow \text{Mod}(M) \longrightarrow 1, \quad (5)$$

where $\overset{\circ}{M}$ denotes the interior of M . For a proof of the existence of this Birman exact sequence see [JR11, Section 6].

Let $\mathbf{p} = (p_1, \dots, p_n) \in \text{Conf}_n(M)$ be a fixed base point. Since $d \geq 3$, then from [Bir69, Theorem 1] it follows that the fundamental group

$$\pi_1(\text{Conf}_n(\overset{\circ}{M}), \mathbf{p}) \approx \pi_1(\text{Conf}_n(M), \mathbf{p}) \approx \pi_1(M^n, \mathbf{p}) \approx \prod_{i=1}^n \pi_1(M, p_i).$$

Hence the FI-module $H^q(\pi_1(\text{Conf}_\bullet(\overset{\circ}{M})))$ is precisely $H^q(\pi_1(M)^\bullet)$. If the group $\pi_1(M)$ is of type FP_∞ , then from Proposition 4.1 we have that this $\text{FI}[\mathbf{G}]$ -module is finitely generated of weight $\leq q$ with stability degree $\leq q$. From our discussion in Section 5.2 with $\beta = 1$ we can conclude Theorem 1.3.

6.3 The case of manifolds with boundary

When the surface $\Sigma_{g,r}$ in Section 6.1 or the manifold M in Section 6.2 has nonempty boundary, the cohomology of the corresponding pure mapping class groups actually has an $\text{FI}\#$ -module structure.

Proposition 6.1. *Let k be a field or \mathbb{Z} and $i \geq 0$. If M is a connected smooth manifold of dimension $d \geq 2$ with nonempty boundary, then the FI-module $H^i(\text{PMod}^\bullet(M); k)$ has the structure of an $\text{FI}\#$ -module. In particular, $H^i(\text{PMod}^\bullet(M))$ has injectivity degree 0 (when $k = \mathbb{Q}$).*

Proof. We just prove that $\text{PMod}^\bullet(M)$ has the structure of an $\text{FI}\#$ -group. Fix a boundary component ∂_0 of ∂M . For S a finite subset of \mathbb{N} , let us denote by M^S the manifold M with $|S|$ points in the interior of M labeled by the elements in S . For any subset B of S we can consider the manifold with boundary $\partial_0 \times I$ with $|S \setminus B|$ marked points in its interior labeled with the elements in $S \setminus B$, that is $(\partial_0 \times I)^{S \setminus B}$. Then we can define

$$\phi_{B,S} : M^B \hookrightarrow M^B \sqcup_{\partial_0} (\partial_0 \times I)^{S \setminus B} \approx M^S.$$

Now, consider $(A, B, \psi) \in \text{Hom}_{\text{FI}\#}(\mathbf{m}, \mathbf{n})$ where $A \subseteq [m]$, $B \subseteq [n]$ and $\psi : A \rightarrow B$ is a bijection. Let $|_A : M^{[m]} \rightarrow M^A$ be given by forgetting the points labeled with the elements in $[m] \setminus A$. Then we can take the composition

$$(\phi_{B,[n]} \circ \psi \circ |_A) : M^{[m]} \rightarrow M^{[n]}.$$

Above, abusing notation $\psi : M^A \rightarrow M^B$ corresponds to relabeling the points using $\psi : A \rightarrow B$, and $\phi_{B,[n]} : M^B \rightarrow M^{[n]}$ is defined as before. Therefore we have the $\text{FI}\#$ -group $\text{PMod}^\bullet(M)$ given by $\mathbf{n} \mapsto \text{PMod}^n(M)$ and

$$(A, B, \phi)_* := (\phi_{B,[n]} \circ \psi \circ |_A)_* : \text{PMod}^m(M) \rightarrow \text{PMod}^n(M).$$

The conclusion of the proposition then follows. ■

When a Birman sequence exists (hypothesis of Theorems 1.2 and 1.3), we do have a co-FI#-group extension of $\text{Mod}(M)$. Moreover, Theorem 4.3 states the finite generation of the cohomology of configuration spaces of manifolds with boundary. Hence, the argument at the end of Section 5 implies the following results.

Theorem 6.2. *Let k be any field or \mathbb{Z} . For any $i \geq 0$, $2g + r > 2$ and $r > 0$ the FI-module $H^i(\text{PMod}_{g,r}^\bullet; k)$ has the structure of an FI#-module which is finitely generated in degree $\leq 2i$.*

Theorem 6.3. *Let k be any field or \mathbb{Z} . Let M be a smooth connected manifold of dimension $d \geq 3$ with non-empty boundary that satisfies the hypotheses of Theorem 1.3. Then, for any $i \geq 0$, the FI-module $H^i(\text{PMod}^\bullet(M); k)$ has the structure of an FI#-module that is finitely generated in degree $\leq 2i$.*

From the classification of FI#-modules given in [CEF, Theorem 2.24] and the cases when k is either \mathbb{Z} or the fields \mathbb{Q} or $\mathbb{Z}/p\mathbb{Z}$ in Theorems 6.2 and 6.3, we obtain Theorems 1.6 and 1.7, respectively.

6.4 Cohomology of classifying spaces for some diffeomorphism groups

Let M be a connected and compact smooth manifold of dimension $d \geq 3$. We have a fiber bundle

$$B \text{PDiff}^n(M) \rightarrow B \text{Diff}(M \text{ rel } \partial M) \quad (6)$$

where the “fiber” is given by $\text{Diff}(M \text{ rel } \partial M)/\text{PDiff}^n(M) \approx \text{Conf}_n(\mathring{M})$, the configuration space of n ordered points in \mathring{M} , the interior of M . This gives us a functor from \mathbf{FI}^{op} to the category $\mathbf{Fib}(B \text{Diff}(M \text{ rel } \partial M))$. The hypotheses of Theorem 1.4 give the setting needed to apply the arguments in Section 5.2 with $\beta = 1$ to get the desired conclusion.

7 Application to cohomology of some wreath products

Let G be a group of type FP_∞ . The wreath product $G \wr S_n$ is the semidirect product $G^n \rtimes S_n$, where S_n acts on G^n by permuting the coordinates. Therefore there is a split short exact sequence

$$1 \rightarrow G^n \rightarrow G \wr S_n \rightarrow S_n \rightarrow 1$$

For any $i \geq 0$ and any partition λ , a transfer argument (see for example [CF, Corollary 4.4]) implies that the dimension of $H^i(G \wr S_n; V(\lambda)_n)$ is equal to the multiplicity of $V(\lambda)_n$ in $H^i(G^n; \mathbb{Q})$. But this multiplicity is constant for $n \geq 2i$ (see Table 1 in the introduction). Hence we obtain cohomological stability for the group $G \wr S_n$ with coefficients in any S_n -representation for any $n \geq 2i$.

More generally, let PK be a co-FI-group given by $\mathbf{n} \mapsto PK_n$. Assume that there is a sequence of groups K_n such that, for each n , we have the following short exact sequence:

$$1 \rightarrow PK_n \rightarrow K_n \rightarrow S_n \rightarrow 1$$

The wreath product $G \wr K_n$ is the semidirect product $G^n \rtimes K_n$, where K_n acts on G^n via the surjection $K_n \rightarrow S_n$. Therefore there is a split short exact sequence

$$1 \rightarrow G^n \times PK_n \rightarrow G \wr K_n \rightarrow S_n \rightarrow 1$$

On the other hand, for any $i \geq 0$, the naturality of the Künneth formula implies the following isomorphism of FI-modules:

$$H^i(G^\bullet \times PK) = \bigoplus_{p+q=i} H^p(G^\bullet) \otimes H^q(PK).$$

Suppose that the graded FI-module $H^*(PK)$ is known to be of finite type. In [CEF, Proposition 2.61] is proved that finite generation is closed under tensor products, therefore the FI-modules $H^p(G^\bullet) \otimes H^q(PK)$ are finitely generated for $p, q \geq 0$ such that $p + q = i$. Moreover,

$$\text{weight}(H^p(G^\bullet) \otimes H^q(PK)) \leq \text{weight}(H^p(G^\bullet)) + \text{weight}(H^q(PK)).$$

It follows that the consistent sequence $H^i(G^n \times PK_n; \mathbb{Q})$ is monotone and uniformly representation stable (although we do not always get a specific stable range).

As before, the dimension of $H^i(G \wr K_n; V(\lambda)_n)$ is given by the multiplicity of $V(\lambda)_n$ in $H^i(G^n \times PK_n; \mathbb{Q})$, which is eventually constant by uniform representation stability. Therefore we have that $H^i(G \wr K_n; V(\lambda)_n) \approx H^i(G \wr K_{n+1}; V(\lambda)_n)$ for any n sufficiently large. In particular, we obtain rational homological stability for the groups $G \wr K_n$.

Parts (ii), (iii) and (iv) of Theorem 1.10 follow from applying the above discussion to the short exact sequences:

$$\begin{aligned} 1 &\rightarrow P_n(\Sigma_{g,r}) \rightarrow B_n(\Sigma_{g,r}) \rightarrow S_n \rightarrow 1 \\ 1 &\rightarrow \text{PMod}_{g,r}^n \rightarrow \text{Mod}_{g,r}^n \rightarrow S_n \rightarrow 1 \\ 1 &\rightarrow \text{PMod}^n(M) \rightarrow \text{Mod}^n(M) \rightarrow S_n \rightarrow 1 \end{aligned}$$

To obtain Theorem 1.10 part (v), we consider the co-FI-groups $\text{P}\Sigma_\bullet$ and Σ_\bullet^+ , which are functors from \mathbf{FI}^{op} to \mathbf{Gp} given by $\mathbf{n} \mapsto \text{P}\Sigma_n$, the *pure string motion group* (see definition in [Wil12, Sections 1 & 2]) and $\mathbf{n} \mapsto \Sigma_n^+$, the braid permutation group, respectively. In [Wil12, Theorem 6.4] Wilson proved that for any $k \geq 0$ the sequence $\{H^k(\text{P}\Sigma_n; \mathbb{Q})\}$ satisfies uniform representation stability with stable range $n \geq 4k$. Therefore, [CEF, Theorem 1.14] implies that for any $k \geq 0$ the FI-module $H^k(\text{P}\Sigma_\bullet)$ is finitely generated.

The co-FI-groups Σ_\bullet^+ and $\text{P}\Sigma_\bullet$ are related by the following short exact sequence:

$$1 \rightarrow \text{P}\Sigma_n \rightarrow \Sigma_n^+ \rightarrow S_n \rightarrow 1,$$

which give us again the setting discussed above.

References

- [AC96] E. Arbarello and M. Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves*, J. Algebraic Geom. **5** (1996), no. 4, 705–749.
- [Bir69] J. S. Birman, *On braid groups*, Comm. Pure Appl. Math. **22** (1969), 41–72.
- [Bir74] ———, *Braids, links, and mapping class groups*, Annals of Mathematics Studies, no. 82, Princeton University Press, Princeton, N.J., 1974.
- [Bro94] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original.
- [CEF] T. Church, J. Ellenberg, and B. Farb, *FI-modules: a new approach for S_n -representations*, arXiv:1204.4533.
- [CF] T. Church and B. Farb, *Representation theory and homological stability*, arXiv:1008.1368.

- [Chu] T. Church, *Homological stability for configuration spaces of manifolds*, to appear in Invent. Math.
- [FH91] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics.
- [FM12] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.
- [FP] C. Faber and R. Pandharipande, *Tautological and non-tautological cohomology of the moduli space of curves*, to appear in the Handbook of Moduli.
- [Har85] J. L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2) **121** (1985), no. 2, 215–249.
- [Har88] ———, *The cohomology of the moduli space of curves*, Theory of moduli (Montecatini Terme, 1985), Lecture Notes in Math., vol. 1337, Springer, Berlin, 1988, pp. 138–221.
- [Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
- [HL97] R. Hain and E. Looijenga, *Mapping class groups and moduli spaces of curves*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 97–142.
- [HM97] A. Hatcher and D. McCullough, *Finiteness of classifying spaces of relative diffeomorphism groups of 3-manifolds*, Geom. Topol. **1** (1997), 91–109 (electronic).
- [HW10] A. Hatcher and N. Wahl, *Stabilization for mapping class groups of 3-manifolds*, Duke Math. J. **155** (2010), no. 2, 205–269.
- [JR11] R. Jimenez Rolland, *Representation stability for the cohomology of the moduli space \mathcal{M}_g^n* , Algebraic & Geometric Topology **11** (2011), no. 5, 3011–3041.
- [KV07] J. Kock and I. Vainsencher, *An invitation to quantum cohomology*, Progr. Math., vol. 249, Birkhäuser, Basel, 2007.
- [Put] A. Putman, *Representation stability, congruence subgroups, and mapping class groups*, arXiv:1201.487.
- [RW] O. Randal-Williams, *Homological stability for unordered configuration spaces*, to appear in the Quarterly Journal of Mathematics.
- [Tot96] B. Totaro, *Configuration spaces of algebraic varieties*, Topology **35** (1996), no. 4, 1057–1067.
- [Wah] N. Wahl, *Homological stability for mapping class groups of surfaces*, Preprint. arXiv:1006.4476.
- [Wil12] J. Wilson, *Representation stability for the cohomology of the pure string motion groups*, Algebraic & Geometric Topology **12** (2012), no. 2, 909–931.

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